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Remarks on the number operator for generalized quons

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Abstract. We construct the number operator for particles obeying infinite statistics, defined by a generalized q-deformation of the Heisenberg algebra, and prove the positivity of the norm of linearly independent state vectors.

The approach to particle statistics based on deformations of the bilinear Bose and Fermi commutation relations has attracted considerable interest during the last few years [1–4]. The particles obeying this type of statistics are called 'quons'. The quon algebra (or the q-mutator) is given by

$$a_i a_j^\dagger - q a_j^\dagger a_i = \delta_{ij} \qquad (\forall i, j) \tag{1}$$

and interpolates between Bose and Fermi algebras as the deformation parameter q goes from 1 to -1 on the real axis. When supplemented by the vacuum condition

$$a_i|0\rangle = 0 \qquad (\forall i) \tag{2}$$

the quon algebra determines a (Fock-like) representation in a linear vector space. For $q \in [-1, 1]$, the squared norms of all vectors made by the limits of the polynomials of the creation operators $a_k^\dagger$ are strictly positive. No commutation relation can be imposed on $a_i a_j$ or $a_i^\dagger a_j^\dagger$. Furthermore, no such rule is needed to calculate the vacuum matrix elements of the polynomials in the a's and $a^\dagger$'s. All such matrix elements can be calculated by moving the annihilation operators to the right using (1), until they, according to (2), annihilate the vacuum [5].

This paper is a comment on a few previous papers [1, 2, 5–7] and its aim is to construct the number operator for a generalized q-deformation of the Heisenberg algebra characterized by the following relations:

$$a_i a_j^\dagger - q_{ij} a_j^\dagger a_i = \delta_{ij} \qquad (\forall i, j) \tag{3}$$

$$a_i|0\rangle = 0 \qquad (\forall i) \tag{4}$$

$$q_{ji}^* = q_{ij} \tag{5}$$

with the deformations parameters q_{ij} being, in general, complex numbers [8]. The statistics based on the commutation relations (3) generalize classical Bose and Fermi statistics, which

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correspond to $q_{ij} = 1$ $(\forall i, j)$ and $q_{ij} = -1$ $(\forall i, j)$, respectively. Relation (5) follows from the consistency requirement of relation (3).

For each k, the kth number operator N_k satisfies the relations

$$\begin{aligned} &N_k^\dagger = N_k \qquad N_k|0\rangle = 0 \\ &[N_k, a_l^\dagger] = \delta_{kl} a_l^\dagger \qquad (\forall l) \end{aligned} \tag{6}$$

which is equivalent to

$$N_k(a_{i_1}^\dagger a_{i_2}^\dagger \cdots a_{i_n}^\dagger |0\rangle) = s(a_{i_1}^\dagger a_{i_2}^\dagger \cdots a_{i_n}^\dagger |0\rangle) \tag{7}$$

where s is the index number i_j, such that $i_j = k$. The most general expression for the number operator N_k is of the form

$$\begin{aligned} N_k = a_k^\dagger a_k + \sum_{n=2}^{\infty} \sum_{(\boldsymbol{i}_{n-1})} \sum_{\pi(k,\boldsymbol{i}_{n-1})} \sum_{\sigma(k,\boldsymbol{i}_{n-1})} c_{\pi(k,\boldsymbol{i}_{n-1}),\sigma(k,\boldsymbol{i}_{n-1})} \cdot a_{\pi(k)}^\dagger a_{\pi(i_1)}^\dagger \cdots \\ a_{\pi(i_{n-1})}^\dagger a_{\sigma(i_{n-1})} \cdots a_{\sigma(i_1)} a_{\sigma(k)} \end{aligned} \tag{8}$$

where $\boldsymbol{i}_{n-1} \equiv (i_1, \ldots, i_{n-1})$ is an arbitrary choice of $n-1$ indices, *including their repetitions*, whereas π, σ are permutations of n indices. Making use of relation (7), one finds that the condition satisfied by the coefficients $c_{i_1,\ldots,i_n;j_1,\ldots,j_n}$ can be written in the form of the following matrix equation:

$$[(c_{i_1,\ldots,i_n;j_1,\ldots,j_n})] \times M_n(q) = Q_n(q). \tag{9}$$

The matrix M_n is defined by

$$(M_n)_{i_1,\ldots,i_n;j_1,\ldots,j_n} = \langle 0|a_{i_n} \cdots a_{i_1} a_{j_1}^\dagger \cdots a_{j_n}^\dagger|0\rangle. \tag{10}$$

Taking into account equations (3)–(5) and (10) and using the method of mathematical induction, one arrives at the closed-form expression

$$(M_n)_{\pi(1,\ldots,n);\sigma(1,\ldots,n)} = \prod_{r,s=1}^{n} q_{\pi(r)\sigma(s)}^{P(r,s)} \qquad (\pi(r) \neq \sigma(s)) \tag{11}$$

where

$$P(r,s) = \theta[-(r-s)((\sigma^{-1}\cdot\pi)(r) - (\sigma^{-1}\cdot\pi)(s))] \tag{12}$$

and $\theta(x)$ designates the function defined by

$$\theta(x) = \begin{cases} 1 & \text{if } x > 0 \\ 0 & \text{if } x \leqslant 0. \end{cases} \tag{13}$$

The determinant of the matrix M_n is

$$\det M_n = \prod_{k=1}^{n-1} \left[\prod_{\{i_1,\ldots,i_{k+1}\}} \left(1 - \prod_{\{i_\alpha i_\beta\}} |q_{i_\alpha i_\beta}|^2\right)\right]^{(k-1)!(n-k)!} \tag{14}$$

where the set $\{i_1, \ldots, i_{k+1}\}$ denotes a choice of $k+1$ different indices out of n such indices, whereas $\{i_\alpha i_\beta\}$ is any of its subsets. Unlike in the case of the matrix $M_n(q)$, explicitly given by equations (11) and (12), the closed-form expression for the matrix $Q_n(q)$ cannot be written down. In fact, this matrix is obtained as a result of the action of the lower-order terms $a_{i_1}^\dagger \cdots a_{i_s}^\dagger a_{j_1} \cdots a_{j_s}$, $s < n$, entering expression (8) for the number operator on the eigenvector $a_{k_1}^\dagger \cdots a_{k_n}^\dagger |0\rangle$. Thus, to completely determine the matrix $Q_n(q)$, knowledge of all the lower-order coefficients $c_{i_1,\ldots,i_s;j_1,\ldots,j_s}$, $s = 2, 3, \ldots, n-1$ is required.

As a special case of general formula (14), we give the expression for the determinant of the matrix M_4, corresponding to $n = 4$ and $(k_1, k_2, k_3, k_4) \equiv (k, l, m, p)$:

$$\begin{aligned}\det M_4 = {} & (1 - |q_{kl}|^2)^6 (1 - |q_{km}|^2)^6 (1 - |q_{kp}|^2)^6 (1 - |q_{lm}|^2)^6 (1 - |q_{lp}|^2)^6 (1 - |q_{mp}|^2)^6 \\ & \times (1 - |q_{kl}|^2 |q_{km}|^2 |q_{lm}|^2)^2 (1 - |q_{kl}|^2 |q_{kp}|^2 |q_{lp}|^2)^2 (1 - |q_{km}|^2 |q_{kp}|^2 |q_{lp}|^2)^2 \\ & \times (1 - |q_{lm}|^2 |q_{lp}|^2 |q_{mp}|^2)^2 (1 - |q_{kl}|^2 |q_{km}|^2 |q_{kp}|^2 |q_{lm}|^2 |q_{lp}|^2 |q_{mp}|^2).\end{aligned} \tag{15}$$

It is evident from equation (14), and especially from the special case (15), that if the deformation parameters are such that $|q_{ij}|\langle 1$ $(\forall i, j)$, then the matrix $M_n(q)$ is regular and positively definite. Consequently, the coefficients $c_{i_1,\ldots,i_n;j_1,\ldots,j_n}$, appearing in expression (8) for the number operator N_k, exist and the norm of the state vector $a_{i_1}^\dagger a_{i_2}^\dagger \cdots a_{i_n}^\dagger |0\rangle$ is positive. That being the case, one is now allowed to rewrite equation (9) in the form

$$C_n(q) = Q_n(q) \times [M_n(q)]^{-1} \tag{16}$$

where, for notational simplicity,

$$C_n(q) = [(c_{i_1,\ldots,i_n;j_1,\ldots,j_n})]. \tag{17}$$

Equation (16) represents the main result of this paper.

As an example, in the following we illustrate the calculation of the coefficients $c_{i_1,i_2,i_3;j_1,j_2,j_3}$, which amounts to finding the matrix coefficients $C_3(q)$. According to (16), this matrix is given by

$$C_3(q) = Q_3(q) \times [M_3(q)]^{-1}. \tag{18}$$

There are four different cases to be considered.

Case 1: $n = 3$, $(k_1, k_2, k_3) \equiv (k, k, k)$. This case is trivial, since $Q_3(q)$ and $M_3(q)$ are represented by the numbers (1×1 square matrices)

$$(Q_3(q))_{kkk,kkk} = (1 - q_{kk})^2 (1 + q_{kk}) \tag{19}$$

$$(M_3(q))_{kkk,kkk} = (1 + q_{kk})(1 + q_{kk} + q_{kk}^2) \tag{20}$$

so that, in view of (18),

$$c_{kkk,kkk} = \frac{(1 - q_{kk})^2}{1 + q_{kk} + q_{kk}^2}. \tag{21}$$

Case 2: $n = 3$, $(k_1, k_2, k_3) \equiv (k, k, l)$. In this case, $Q_3(q)$ and $M_3(q)$ are 3×3 square matrices, the rows and columns of which are indexed in the order (k, k, l), (k, l, k) and (l, k, k). The matrix $M_3(q)$ is given by

$$M_3(q) = \begin{bmatrix} 1 + q_{kk} & q_{kl}(1 + q_{kk}) & q_{kl}^2(1 + q_{kk}) \\ q_{lk}(1 + q_{kk}) & 1 + q_{kk}|q_{kl}|^2 & q_{kl}(1 + q_{kk}) \\ q_{lk}^2(1 + q_{kk}) & q_{lk}(1 + q_{kk}) & (1 + q_{kk}) \end{bmatrix} \tag{22}$$

with the determinant

$$\det M_3 = (1 + q_{kk})^2(1 - |q_{kl}|^2)^2(1 - q_{kk}|q_{kl}|^2). \tag{23}$$

On the basis of equations (22) and (23), the inverse matrix of $M_3(q)$ is found to be

$$[M_3(q)]^{-1} = \frac{1}{(1 + q_{kk})(1 - q_{kk}|q_{kl}|^2)} \times \begin{bmatrix} 1 & -q_{kl}(1 + q_{kk}) & q_{kk}q_{kl}^2 \\ -q_{lk}(1 + q_{kk}) & (1 + q_{kk})(1 + |q_{kl}|^2) & -q_{kl}(1 + q_{kk}) \\ q_{kk}q_{lk}^2 & -q_{lk}(1 + q_{kk}) & 1 \end{bmatrix}. \tag{24}$$

The matrix Q_3 can be obtained using the following lower-order coefficients:

$$\begin{aligned} c_{kk,kk} &= \frac{1 - q_{kk}}{1 + q_{kk}} \qquad c_{kl,kl} = -\frac{q_{kl}}{1 - |q_{kl}|^2} \qquad c_{lk,kl} = \frac{1}{1 - |q_{kl}|^2} \\ c_{kl,lk} &= \frac{|q_{kl}|^2}{1 - |q_{kl}|^2} \qquad c_{lk,kl} = \frac{-q_{lk}}{1 - |q_{kl}|^2}. \end{aligned} \tag{25}$$

The result is

$$Q_3(q) = \begin{bmatrix} 0 & -q_{kl} & -q_{kl}^2(1 - q_{kk}) \\ 0 & 1 + q_{kk}|q_{kl}|^2 & 0 \\ 0 & -q_{kk}q_{lk} & -1 + q_{kk} \end{bmatrix}. \tag{26}$$

The coefficient matrix $C_3(q)$ is now obtained by substituting equations (24) and (26) into equation (18). To see what the structure of these coefficients is like, we only give the elements of the first row of this matrix:

$$\begin{aligned} c_{kkl,kkl} &= 2q_{kk}q_{kl}^2 \cdot \Delta \\ c_{kkl,klk} &= -q_{kl}(1 + q_{kk})(1 + q_{kk}|q_{kl}|^2) \cdot \Delta \\ c_{kkl,lkk} &= |q_{kl}|^2(1 + q_{kk} - q_{kk}|q_{kl}|^2 + q_{kk}{}^2|q_{kl}|^2) \cdot \Delta \end{aligned} \tag{27}$$

with

$$\Delta = \frac{1}{(1 + q_{kk})(1 - q_{kk}|q_{kl}|^2)}. \tag{28}$$

Case 3: $n = 3$, $(k_1, k_2, k_3) \equiv (k, l, l)$. In this case, the results are of the same structure as those in the preceding case, so we do not give them here.

Case 4: $n = 4$, $(k_1, k_2, k_3) \equiv (k, l, m)$. In this case, $Q_3(q)$ and $M_3(q)$ are 6×6 square matrices, the rows and columns of which are indexed by the elements of the permutation group in the order (k, l, m), (l, k, m), (k, m, l), (l, m, k), (m, k, l) and (m, l, k). Again, the matrix $Q_3(q)$ can be obtained with the help of the lower-order coefficients (25), and is given by

$$Q_3(q) = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & q_{kl}q_{km}q_{lm} \\ 0 & 0 & 0 & -q_{km} & 0 & -q_{km}q_{lm} \\ 0 & 0 & 0 & q_{kl}q_{km}q_{ml} & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -q_{kl}q_{ml} & 0 & -q_{kl} \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}. \tag{29}$$

On the basis of equation (11), the matrix $M_3(q)$ is found to be

$$M_3(q) = \begin{bmatrix} 1 & q_{kl} & q_{lm} & q_{kl}q_{km} & q_{km}q_{lm} & q_{kl}q_{km}q_{lm} \\ q_{lk} & 1 & q_{lk}q_{lm} & q_{km} & q_{lm}q_{lk}q_{km} & q_{lm}q_{km} \\ q_{ml} & q_{kl}q_{ml} & 1 & q_{kl}q_{km}q_{ml} & q_{km} & q_{km}q_{kl} \\ q_{lk}q_{mk} & q_{mk} & q_{lk}q_{lm}q_{mk} & 1 & q_{lm}q_{lk} & q_{lm} \\ q_{mk}q_{ml} & q_{ml}q_{mk}q_{kl} & q_{mk} & q_{ml}q_{kl} & 1 & q_{kl} \\ q_{mk}q_{ml}q_{lk} & q_{ml}q_{mk} & q_{mk}q_{lk} & q_{ml} & q_{lk} & 1 \end{bmatrix} \tag{30}$$

with the determinant

$$\det M_3(q) = (1 - |q_{kl}|^2)^2(1 - |q_{km}|^2)^2(1 - |q_{lm}|^2)^2(1 - |q_{kl}|^2|q_{km}|^2|q_{lm}|^2). \tag{31}$$

Inserting the matrices $Q_3(q)$, given by equation (29), and the matrix $M_3^{-1}(q)$, obtainable from equations (30) and (31), into equation (18), one finds the coefficient matrix $C_3(q)$. Here, as in case 2, we only exibit the elements comprising its first row. They are

$$\begin{aligned} c_{klm,klm} &= (M_3^{-1})_{klm,mlk} \\ c_{klm,lkm} &= (M_3^{-1})_{lkm,mlk} \\ c_{klm,kml} &= (M_3^{-1})_{kml,mlk} \\ c_{klm,lmk} &= (M_3^{-1})_{lmk,mlk}|q_{kl}|^2|q_{km}|^2|q_{lm}|^2 \\ c_{klm,mkl} &= (M_3^{-1})_{mkl,mlk}|q_{kl}|^2|q_{km}|^2|q_{lm}|^2 \\ c_{klm,mlk} &= (M_3^{-1})_{mlk,mlk}|q_{kl}|^2|q_{km}|^2|q_{lm}|^2 \end{aligned} \tag{32}$$

where

$$\begin{aligned} (M_3^{-1})_{klm,mlk} &= q_{kl}q_{km}q_{lm}(1 - |q_{km}|^2)(1 - |q_{kl}|^2|q_{lm}|^2) \\ (M_3^{-1})_{lkm,mlk} &= -|q_{kl}|^2q_{km}q_{lm}(1 - |q_{km}|^2)(1 - |q_{lm}|^2) \\ (M_3^{-1})_{kml,mlk} &= -q_{kl}q_{km}|q_{lm}|^2(1 - |q_{kl}|^2)(1 - |q_{km}|^2) \\ (M_3^{-1})_{lmk,mlk} &= -q_{lm}(1 - |q_{kl}|^2)(1 - |q_{km}|^2) \\ (M_3^{-1})_{mkl,mlk} &= -q_{kl}(1 - |q_{km}|^2)(1 - |q_{lm}|^2) \\ (M_3^{-1})_{mlk,mlk} &= (1 - |q_{km}|^2)(1 - |q_{kl}|^2|q_{lm}|^2) \end{aligned} \tag{33}$$

are the relevant matrix elements of $M_3^{-1}(q)$.

Next we derive a relation that will make it possible to rewrite expression (8), for the number operator N_k, in a more compact and elegant form.

Making use of equations (11) and (25), one finds that, for the case $n = 2$ and $(k_1, k_2) \equiv (k, l)$, the following relation holds:

$$\sum_{\pi(k,l)} \sum_{\sigma(k,l)} c_{\pi(k,l),\sigma(k,l)} a^{\dagger}_{\pi(k)} a^{\dagger}_{\pi(l)} a_{\sigma(k)} a_{\sigma(l)} = (M_2^{-1})_{kl,kl} \tilde{a}^{\dagger}_{kl} \tilde{a}_{kl}. \tag{34}$$

For the case $n = 3$ and $(k_1, k_2, k_3) \equiv (k, l, m)$, an analogous relation can be established on the basis of equations (32) and (33). It reads:

$$\begin{aligned} &\sum_{\pi(k,l,m)} \sum_{\sigma(k,l,m)} c_{\pi(k,l,m),\sigma(k,l,m)} a^{\dagger}_{\pi(k)} a^{\dagger}_{\pi(l)} a^{\dagger}_{\pi(m)} a_{\sigma(k)} a_{\sigma(l)} a_{\sigma(m)} \\ &\qquad = \sum_{\pi(l,m)} \sum_{\sigma(l,m)} (M_3^{-1})_{k,\sigma(l,m);k,\pi(l,m)} \tilde{a}^{\dagger}_{k,\pi(l,m)} \tilde{a}_{k,\sigma(l,m)} \end{aligned} \tag{35}$$

with the notation

$$\tilde{a}_{k,l} = a_k a_l - q_{lk} a_l a_k \tag{36}$$

$$\tilde{a}_{k,l,m} = \tilde{a}_{k,l} a_m - q_{mk} q_{ml} a_m \tilde{a}_{k,l}. \tag{37}$$

Generalizing equations (34) and (35) for arbitrary n, one arrives at the following relation:

$$\begin{aligned} &\sum_{\pi(k,i_{n-1})} \sum_{\sigma(k,i_{n-1})} c_{\pi(k,i_{n-1}),\sigma(k,i_{n-1})} a^{\dagger}_{\pi(k)} a^{\dagger}_{\pi(i_1)} \cdots a^{\dagger}_{\pi(i_{n-1})} a_{\sigma(k)} a_{\sigma(i_1)} \cdots a_{\sigma(i_{n-1})} \\ &\qquad = \sum_{\pi(i_{n-1})} \sum_{\sigma(i_{n-1})} (M_n^{-1})_{k,\sigma(i_{n-1});k,\pi(i_{n-1})} \tilde{a}_{k,\pi(i_{n-1})})^{\dagger} \tilde{a}_{k,\sigma(i_{n-1})} \end{aligned} \tag{38}$$

where

$$\tilde{a}_{k,j_1,\ldots,j_{n-1}} = \tilde{a}_{k,j_1,\ldots,j_{n-2}} a_{j_{n-1}} - q_{j_{n-1}k} q_{j_{n-1}j_1} \cdots q_{j_{n-1}j_{n-2}} a_{j_{n-1}} \tilde{a}_{k,j_1,\ldots,j_{n-2}}. \tag{39}$$

Apart from the elegance of equations (34), (35) and (38), the expression for the number operator makes it easier to relate our results to the results of Greenberg [2], corresponding to the $q_{ij} = 0$ $(\forall i, j)$ statistics.

Basically, our result represents a generalization of the previous work performed by Zagier [6], to the case where q_{ij} depends on i, j. It is important to note the paper by Stanciu [6] in which Zagier's statement, about the conjecture for the number operator in the case where $q_{ij} = q$ is independent of i, j, is proved.

All the results obtained above correspond to the case where the deformation parameters satisfy the condition

$$|q_{ij}| < 1 \qquad (\forall i, j). \tag{40}$$

Before concluding, we briefly discuss the cases where we have the following conditions instead of (41):

$$|q_{ij}| > 1 \qquad (\forall i, j) \tag{41}$$

or

$$|q_{ij}| = 1 \qquad \text{i.e. } q_{ij} = \mathrm{e}^{\mathrm{i}\Phi_{ij}} \qquad (\forall i, j). \tag{42}$$

If the case (42) is realized, the number operator exists. This is clearly seen from equations (14) and (16). However, as is evident, for example, from equations (15), (23) and (31), the norm of the state vector $a_{i_1}^\dagger a_{i_2}^\dagger \cdots a_{i_n}^\dagger |0\rangle$ is not positive, making this case physically unacceptable.

However, if the q_{ij} are such that condition (43) is satisfied, even though the matrix $M_n(q)$ is singular, the existence of the number opearator is not excluded. In this case, the existence of the operator N_k depends on whether or not it is possible to impose such a q-mutator on the operator pairs a_i, a_j $(\forall i, j)$ that all terms higher than $a_k^\dagger a_k$ cancel. This is precisely what happens for $q_{ij} = \pm 1$, corresponding to Bose and Fermi statistics, respectively. Further investigation regarding this point is unquestionably of interest and is the subject of another study. For the statistics characterized by equations (43), it should be pointed out that anyons (particles existing in $2+1$ dimensions) can alternatively be represented as a q-deformation of an underlying bosonic algebra. This can be viewed as an extension of Greenberg's approach with q being a complex number $|q| = 1$ [7].

To conclude: in this paper we have studied the generalized q-deformation of the Heisenberg algebra defined by equations (3)–(5). For the case when deformation parameters are such that $|q_{ij}| < 1 (\forall i, j)$ we have proved the existence and presented a method for explicit construction of the number operator for particles obeying the corresponding statistics. We have also proved the positivity of the norm of linearly-independent state vectors.

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References

[1] Greenberg O W 1990 *Bull. Am. Phys. Soc.* **35** 981
[2] Greenberg O W 1990 *Phys. Rev. Lett.* **64** 705
[3] Mohapatra R N 1990 *Phys. Lett.* **242B** 407
[4] Fivel D I 1990 *Phys. Rev. Lett.* **65** 3361
[5] Greenberg O W 1991 *Phys. Rev.* D **43** 4111
[6] Zagier D 1992 *Commun. Math. Phys.* **147** 199
Stanciu S 1992 *Commun. Math. Phys.* **147** 211
[7] Bardek V, Dorešić M and Meljanac S 1993 Anyons as quon particles *Phys. Rev.* D to appear; 1993 Example of q-deformed field theory *Int. J. Mod. Phys.* to appear
[8] Meljanac S private communication